

Fragmentation Functions of pion, kaon and proton at NLO approximation: Laplace Transform approach

M.Zarei ^{a,*}, F.Taghavi-Shahri ^{a,†}, S. Atashbar Tehrani ^{b,‡} and M.Sarbishei ^{a,§}

^(a) *Department of Physics, Ferdowsi University of Mashhad,*

P.O. Box 1436, Mashhad, Iran

^(b) *Independent researcher,*

P.O. Box 1149-8834413 Tehran, Iran

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Using repeated Laplace transform, We find an analytical solution for DGLAP evolution equations for extracting the pion, kaon and proton Fragmentation Functions (FFs) at NLO approximation. We also study the symmetry breaking of the sea quarks Fragmentation Functions, $D_q^h(z, Q^2)$ and simply separated them according to their mass ratio. Finally, we calculate the total Fragmentation Functions of these hadrons and compare them with experimental data and those from global fits. Our results show a good agreement with the FFs obtained from global parameterizations as well as with the experimental data.

I. INTRODUCTION

Understanding the basic internal structure of matter and the quest for the ultimate constituents has always been important in high energy physics. The nucleons are the basic building blocks of atomic nuclei. The internal structure of the nucleons determines their fundamental properties and directly affect the properties of the nuclei. Therefore, understanding how the nucleon is built in terms of its constituents is an important and challenging question in modern nuclear physics.

Information about nucleon structure comes from two important processes: The first one is semi-inclusive deep inelastic scattering (SIDIS), whose reaction is as follows: $l + N \rightarrow l + h + X$, and the second one is semi-inclusive hadron reaction like: $p + p \rightarrow h + X$. However, both of the processes require a knowledge of the parton fragmentation functions (FFs) which describe the transition parton to hadron: $parton \rightarrow h + X$.

In general, fragmentation is the QCD process in which partons hadronize to colorless hadrons and the fragmentation functions, $D_i^h(z, Q^2)$, represent the probability for a parton i to fragments into a particular hadron h carrying a certain fraction of the parton energy or momentum. They are a necessary ingredient in calculation of the single hadron inclusive production in any processes like $p\bar{p}$, ep , γp and $\gamma\gamma$ scattering.

Fragmentation functions cannot be computed directly from perturbative QCD because, transition between color partons into colorless hadrons is a soft/long-distance process, leading to divergences in the perturbation theory. Perturbative QCD dose not know anything about experimentally measured hadrons, but only quarks, anti-quarks and gluons. Fragmentation Functions can be evolved

with DGLAP evolution equations from a starting distribution at a defined energy scale [1, 2].

Recently we have used Laplace transform and provided an analytical method to calculate Polarized Parton Distribution functions (PPDFs)[3, 4]. In the present paper we will apply this new method introduced by Block et al.[5–10] to calculate pion, kaon and proton fragmentation functions. Therefore, our main task is finding analytical solutions of DGLAP evolution equations to extract Fragmentation Functions (FFs). To do this, we use the Laplace transform and find analytical solution of DGLAP equations for FFs. The initial inputs are selected from HKNS code to warranty the correctness of our analytical calculations. Finally, comparison of our FFs with those from global fits and also with experimental data confirms the validity of our calculations.

The paper is organized as follows. In Section 2 we review the method of analytical solution of DGLAP evolution equations for extracting Fragmentation Functions based on the Laplace transform. Then, in Section 3 we utilize this method to calculate the Fragmentation Functions (FFs) of pion, kaon and proton. We also find a simple scenario for studying the symmetry breaking in the sea quarks FFs. Finally in section 4 we calculated the total fragmentation functions of pion, kaon and proton and also compared them with available experimental data [11] and those from global fits [12–16].

II. ANALYTICAL SOLUTION OF DGLAP EVOLUTION EQUATIONS FOR EXTRACTING FRAGMENTATION FUNCTIONS BASED ON THE LAPLACE TRANSFORMS

The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [17–19], for the Fragmentation Functions (FFs) can be written as follows

* m_zarei_128@yahoo.com

† f_taghavi@ipm.ir

‡ atashbar@ipm.ir

§ sarbishei@um.ac.ir

[2]:

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_{ns}}{\partial \ln(Q^2)}(z, Q^2) = D_{ns} \otimes [P_{qq}^{LO,ns} + \frac{\alpha_s(\tau)}{4\pi} P_{qq}^{NLO,ns}] (z, Q^2). \quad (1)$$

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_s}{\partial \ln Q^2}(z, Q^2) = D_s \otimes \left(P_{qq}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{qq}^1 \right) (z, Q^2) + D_g \otimes \left(P_{gq}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{gq}^1 \right) (z, Q^2), \quad (2)$$

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_g}{\partial \ln Q^2}(z, Q^2) = D_s \otimes \left(P_{qg}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{qg}^1 \right) (z, Q^2) + D_g \otimes \left(P_{gg}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{gg}^1 \right) (z, Q^2). \quad (3)$$

where $P_{ij}^{0,1}$ are the leading and next to leading order splitting functions. Block et al. in Refs.[5–7] showed that using the Laplace transform, one can solve the DGLAP evolution equations directly and extract unpolarized parton distribution functions. It is possible to solve analytically the coupled leading and next-to-leading-order DGLAP evolution equations to extract Fragmentation Functions too. We will give the details here and review the method for extracting the Fragmentation Functions at NLO approximation.

According to Block's scenario, by introducing the variable $\nu \equiv \ln(\frac{1}{z})$ into the coupled DGLAP equations, one can turn them into coupled convolution equations in ν space. Now, using a new variable, namely, $\tau \equiv \frac{1}{4\pi} \int_{Q_0^2}^{Q^2} \alpha_s(Q'^2) d \ln Q'^2$, one can use two Laplace transforms from ν space to s space and from τ space to U space. With these two Laplace transforms, the DGLAP evolution equations can be solved iteratively by a set of convolution integrals which are related to Fragmentation Functions at an initial input scale of Q_0^2 . Finally, two inverse Laplace transformations will back us to the usual space (z, Q^2) [4, 10].

A. Non- Singlet Fragmentation Functions

At the NLO approximation, the fragmentation of valence quarks into hadrons are given by DGLAP evolution equations as:

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_{ns}}{\partial \ln(Q^2)}(z, Q^2) = D_{ns} \otimes [P_{qq}^{LO,ns} + \frac{\alpha_s(\tau)}{4\pi} P_{qq}^{NLO,ns}] (z, Q^2). \quad (4)$$

where

$$D_{ns}^h(z, Q^2) = D_q^h(z, Q^2) - D_{\bar{q}}^h(z, Q^2) \quad (5)$$

The \otimes symbol in the above equations refers to the convolution integral in which the splitting functions in the right-hand side of Eq. (4) are in fact functions of a variable such as $\frac{x}{z}$. Using the new variables $\nu \equiv \ln(\frac{1}{z})$, $w \equiv \ln(\frac{1}{x})$ and $\tau \equiv \frac{1}{4\pi} \int_{Q_0^2}^{Q^2} \alpha_s(Q'^2) d \ln Q'^2$ and also defining of $z D_{ns}(z, Q^2) = F_{ns}(z, Q^2)$, then we have the DGLAP evolution equation as a function of ν and τ variables as

$$\frac{\partial \hat{F}_{ns}}{\partial \tau}(v, \tau) = \int_0^v \hat{F}_{ns}(w, \tau) e^{-(v-w)} [P_{qq}^{LO,ns}(v-w) + \frac{\alpha_s(\tau)}{4\pi} P_{qq}^{NLO,ns}(v-w)] dw. \quad (6)$$

where

$$\hat{F}_{ns}(v, \tau) \equiv F_{ns}(e^{-v}, \tau), \quad (7)$$

Because the r.h.s of Eq. (6) is a normal convolution integral, we can use the following property for the product of Laplace transform :

$$\mathcal{L} \left[\int_0^v \hat{F}[w] \hat{H}[v-w] dw; s \right] = \mathcal{L}[\hat{F}[v]; s] \times \mathcal{L}[\hat{H}[v]; s]. \quad (8)$$

Then we will get a simple solution for valence fragmentation functions in s space:

$$f_{ns}(s, \tau) = e^{\tau \Phi_{ns}(s)} f_{ns0}(s), \quad (9)$$

in which

$$\Phi_{ns}(s) \equiv \Phi_{ns}^{LO}(s) + \frac{\tau_2}{\tau} \Phi_{ns}^{NLO}(s), \quad (10)$$

where

$$\Phi_{ns}^{LO}(s) \equiv \mathcal{L} [e^{-v} P_{qq}^{LO,ns}(e^{-v}); s], \quad \Phi_{ns}^{NLO}(s) \equiv \mathcal{L} [e^{-v} P_{qq}^{NLO,ns}(e^{-v}); s]. \quad (11)$$

The Laplace transform of non- singlet splitting functions, $\Phi_{ns}^{LO}(s)$ and $\Phi_{ns}^{NLO}(s)$ are given in Appendix. A. The τ_2 parameter in Eq. (10) is defined as

$$\tau_2 \equiv \frac{1}{4\pi} \int_0^\tau \alpha_s(\tau') d\tau' = \frac{1}{(4\pi)^2} \int_{Q_0^2}^{Q^2} \alpha_s^2(Q'^2) d \ln Q'^2, \quad (12)$$

In Eq.(9) the $f_{ns0}(s)$ function is the Laplace transform of valence quark fragmentation functions at initial scale of $Q_0^2 = 4.5 \text{ GeV}^2$. We got them from HKNS code [12]. Finally employing the inverse Laplace transform on Eq. (9)[10], we can derive the valence quark fragmentation functions in (z, Q^2) space.

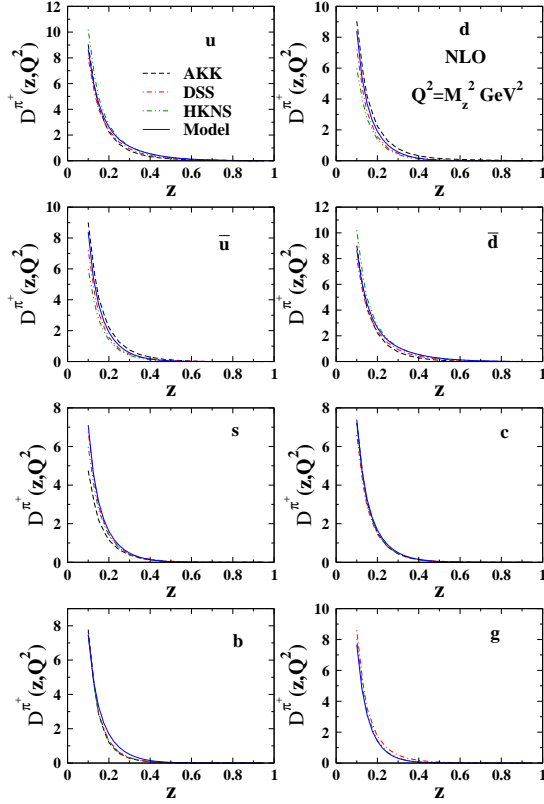


Figure 1: Pion fragmentation functions at $Q^2 = M_z^2$ and Comparison with AKK, DSS and HKNS global fits.

B. Singlet and gluon Fragmentation Functions

The coupled NLO DGLAP evolution equations for extracting the singlet and gluon fragmentation functions are given as follows

$$\begin{aligned} \frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_s}{\partial \ln Q^2}(z, Q^2) &= D_s \otimes \left(P_{qq}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{qq}^1 \right)(z, Q^2) \\ &+ D_g \otimes \left(P_{gq}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{gq}^1 \right)(z, Q^2), \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_g}{\partial \ln Q^2}(z, Q^2) &= D_s \otimes \left(P_{qg}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{qg}^1 \right)(z, Q^2) \\ &+ D_g \otimes \left(P_{gg}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{gg}^1 \right)(z, Q^2). \end{aligned} \quad (14)$$

where the singlet fragmentation function is defined as

$$D_s^h(z, Q^2) = \sum_{q=u,d,s,c,b} [D_q^h(z, Q^2) + D_{\bar{q}}^h(z, Q^2)] \quad (15)$$

Using the convention $zD_s(z, Q^2) \equiv F_s(z, Q^2)$ and $zD_g(z, Q^2) \equiv G(z, Q^2)$, these coupled equations can be

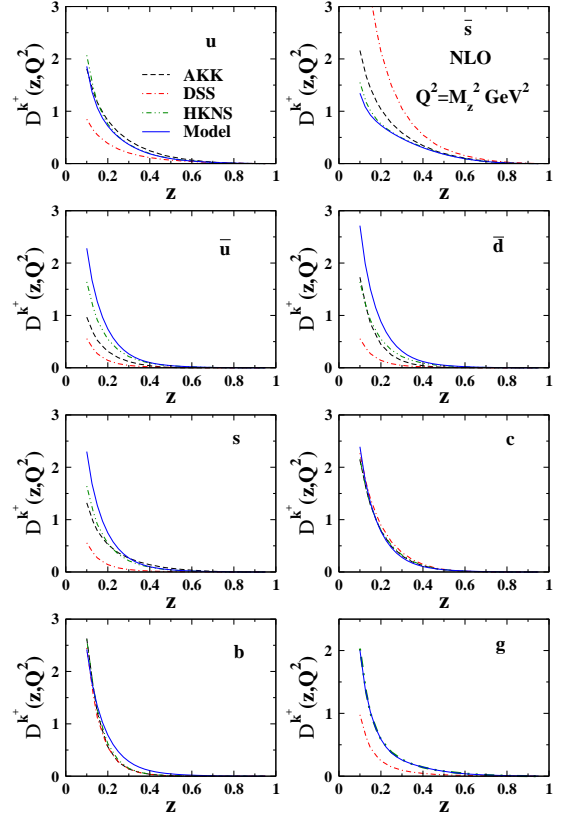


Figure 2: Kaon fragmentation functions at $Q^2 = M_z^2$ and Comparison with AKK, DSS and HKNS global fits.

written in terms of ν and τ variables, which have been defined in previous section. Then we will arrive at:

$$\begin{aligned} \frac{\partial \hat{F}_s}{\partial \tau}(v, \tau) &= \int_0^v \hat{F}_s(w, \tau) \left(\hat{H}_{qq}(v-w) \right. \\ &\quad \left. + \frac{\alpha_s(\tau)}{4\pi} \hat{H}_{qq}^1(v-w) \right) dw \\ &+ \int_0^v \hat{G}(w, \tau) \left(\hat{H}_{gq}(v-w) \right. \\ &\quad \left. + \frac{\alpha_s(\tau)}{4\pi} \hat{H}_{gq}^1(v-w) \right) dw, \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\partial \hat{G}}{\partial \tau}(v, \tau) &= \int_0^v \hat{F}_s(w, \tau) \left(\hat{H}_{qg}(v-w) \right. \\ &\quad \left. + \frac{\alpha_s(\tau)}{4\pi} \hat{H}_{qg}^1(v-w) \right) dw \\ &+ \int_0^v \hat{G}(w, \tau) \left(\hat{H}_{gg}(v-w) \right. \\ &\quad \left. + \frac{\alpha_s(\tau)}{4\pi} \hat{H}_{gg}^1(v-w) \right) dw, \end{aligned} \quad (17)$$

in which we use the definitions:

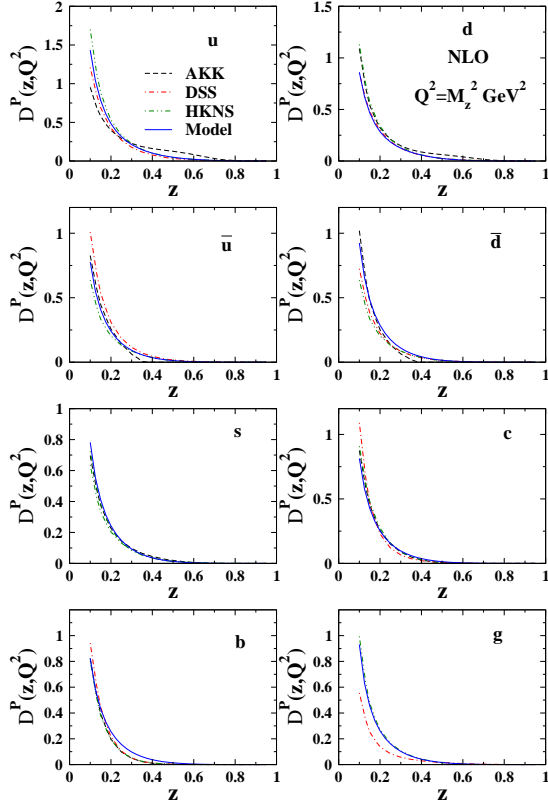


Figure 3: Proton fragmentation functions at $Q^2 = M_z^2$ and Comparison with AKK, DSS and HKNS global fits.

$$\hat{H}_{ij}^0(v) \equiv e^{-v} P_{ij}^0(e^{-v}), \hat{H}_{ij}^1(v) \equiv e^{-v} P_{ij}^1(e^{-v}), \quad (18)$$

$$\hat{F}_s(v, \tau) \equiv F_s(e^{-v}, \tau), \quad \hat{G}(v, \tau) \equiv G(e^{-v}, \tau), \quad (19)$$

At NLO approximation we need two Laplace transforms to decouple the DGLAP equations into two simple equations that can be solved iteratively. The first Laplace transform from ν space to s space changes the DGLAP evolution equation to the first order coupled differential equations as

$$\begin{aligned} \frac{\partial f}{\partial \tau}(s, \tau) &= \left(\Phi_f^{LO}(s) + \frac{\alpha_s(\tau)}{4\pi} \Phi_f^{NLO}(s) \right) f(s, \tau) \\ &+ \left(\Theta_g^{LO}(s) + \frac{\alpha_s(\tau)}{4\pi} \Theta_g^{NLO}(s) \right) g(s, \tau), \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{\partial g}{\partial \tau}(s, \tau) &= \left(\Phi_g^{LO}(s) + \frac{\alpha_s(\tau)}{4\pi} \Phi_g^{NLO}(s) \right) g(s, \tau) \\ &+ \left(\Theta_f^{LO}(s) + \frac{\alpha_s(\tau)}{4\pi} \Theta_f^{NLO}(s) \right) f(s, \tau), \end{aligned} \quad (21)$$

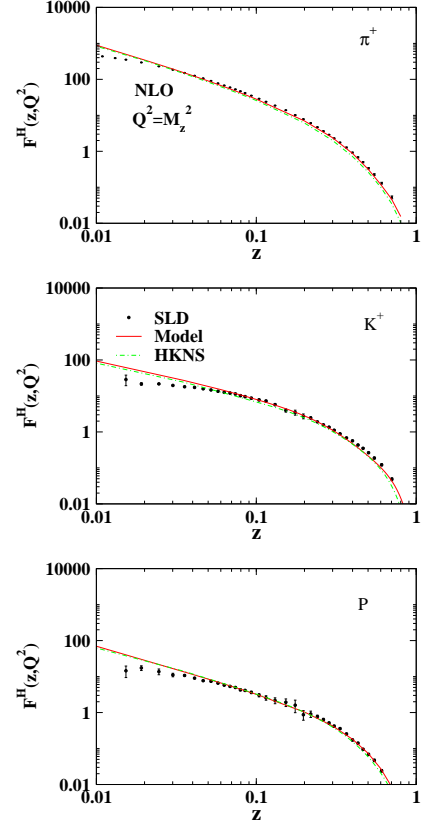


Figure 4: Total fragmentation functions of pion, kaon and proton and comparison with experimental data from SLD [11] at $Q^2 = M_z^2$. We also compared our results with HKNS global fit.

where

$$f(s, \tau) \equiv \mathcal{L}[\hat{F}_s(v, \tau); s], \quad g(s, \tau) \equiv \mathcal{L}[\hat{G}(v, \tau); s] \quad (22)$$

The Laplace transform of singlet and gluon splitting functions, $\Phi_{f,g}^{LO,NLO}(s)$ are given in Appendix. A. The second Laplace transform from τ space to U space changes Eq. (20) and Eq. (21) into two simple linear algebraic equations as

$$\begin{aligned} U\mathcal{F}(s, U) - f_0(s) &= \\ \Phi_f^{LO}(s)\mathcal{F}(s, U) + \Phi_f^{NLO}(s)\mathcal{L}\left[\frac{\alpha_s(\tau)}{4\pi}f(s, \tau); U\right] \\ &+ \Theta_g^{LO}(s)\mathcal{G}(s, U) + \Theta_g^{NLO}(s)\mathcal{L}\left[\frac{\alpha_s(\tau)}{4\pi}g(s, \tau); U\right], \end{aligned} \quad (23)$$

$$\begin{aligned} U\mathcal{G}(s, U) - g_0(s) &= \\ \Phi_g^{LO}(s)\mathcal{G}(s, U) + \Phi_g^{NLO}(s)\mathcal{L}\left[\frac{\alpha_s(\tau)}{4\pi}g(s, \tau); U\right] \\ &+ \Theta_f^{LO}(s)\mathcal{F}(s, U) + \Theta_f^{NLO}(s)\mathcal{L}\left[\frac{\alpha_s(\tau)}{4\pi}f(s, \tau); U\right]. \end{aligned} \quad (24)$$

where we have

$$\mathcal{F}(s, U) \equiv \mathcal{L}[f(s, \tau); U], \quad \mathcal{G}(s, U) \equiv \mathcal{L}[g(s, \tau); U], \quad (25)$$

$$\begin{aligned} \mathcal{L}\left[\frac{\partial f}{\partial \tau}(s, \tau); U\right] &= U\mathcal{F}(s, U) - f_0(s), \\ \mathcal{L}\left[\frac{\partial g}{\partial \tau}(s, \tau); U\right] &= U\mathcal{G}(s, U) - g_0(s), \end{aligned} \quad (26)$$

To simplify the NLO calculations we use an excellent approximation relation $a(\tau) = \frac{\alpha_s(\tau)}{4\pi} \approx a_0 + a_1 e^{-b_1 \tau}$, where $a_0 = 0.0037, a_1 = 0.025, b_1 = 10.7$ [5]. Therefore we write the Laplace transform of $\mathcal{L}[\frac{\alpha_s(\tau)}{4\pi} f(s, \tau); U]$ and $\mathcal{L}[\frac{\alpha_s(\tau)}{4\pi} g(s, \tau); U]$ which are needed in Eq. (24) and Eq. (25) as

$$\begin{aligned} \mathcal{L}\left[\frac{\alpha_s(\tau)}{4\pi} f(s, \tau); U\right] &= \sum_{j=0}^1 a_j \mathcal{F}(s, U + b_j), \\ \mathcal{L}\left[\frac{\alpha_s(\tau)}{4\pi} g(s, \tau); U\right] &= \sum_{j=0}^1 a_j \mathcal{G}(s, U + b_j), b_0 = 0 \end{aligned} \quad (27)$$

Now we define

$$\begin{aligned} \Phi_f(s) &\equiv \Phi_f^{LO}(s) + a_0 \Phi_f^{NLO}(s), \\ \Phi_g(s) &\equiv \Phi_g^{LO}(s) + a_0 \Phi_g^{NLO}(s), \end{aligned} \quad (28)$$

$$\begin{aligned} \Theta_f(s) &\equiv \Theta_f^{LO}(s) + a_0 \Theta_f^{NLO}(s), \\ \Theta_g(s) &\equiv \Theta_g^{LO}(s) + a_0 \Theta_g^{NLO}(s), \end{aligned} \quad (29)$$

Finally, in s and U space, we arrive at the following two coupled algebraic equations for singlet and gluon fragmentation functions which can be solved by iteration method described in [4, 5]:

$$\begin{aligned} [U - \Phi_f(s)] \mathcal{F}(s, U) - \Theta_g(s) \mathcal{G}(s, U) &= f_0(s) \\ + a_1 [\Phi_f^{NLO}(s) \mathcal{F}(s, U + b_1) + \Theta_g^{NLO}(s) \mathcal{G}(s, U + b_1)] &, \end{aligned} \quad (30)$$

$$\begin{aligned} -\Theta_f(s) \mathcal{F}(s, U) + [U - \Phi_g(s)] \mathcal{G}(s, U) &= g_0(s) \\ + a_1 [\Theta_f^{NLO}(s) \mathcal{F}(s, U + b_1) + \Phi_g^{NLO}(s) \mathcal{G}(s, U + b_1)] &, \end{aligned} \quad (31)$$

With the initial input functions for singlet (sum of valence and sea quarks) and gluon sectors of distributions,

which are denoted by $f_0(s)$ and $g_0(s)$ respectively, their evolved solutions in the Laplace s space are given by [9]

$$\begin{aligned} f(s, \tau) &= k_{ff}(s, \tau) f_0(s) + k_{fg}(s, \tau) g_0(s) \\ g(s, \tau) &= k_{gg}(s, \tau) g_0(s) + k_{gf}(s, \tau) f_0(s) \end{aligned} \quad (32)$$

where the k 's in Eq. (32) have been introduced in Refs. [5]. These function are given in Appendix. B for the first iteration. The initial inputs are selected from HKNS code [12] at initial scale of $Q_0^2 = 4.5 \text{ GeV}^2$. Finally with an inverse laplace transform one can derive the singlet and gluon fragmentation functions in (z, Q^2) space [10]. It should be noted that our initial inputs are quoted from HKNS code to confirm the validity of our analytical solutions. If we reach to the acceptable agreement between our FFs and FFs obtained by global fits and also with those from experimental data, then we can be sure that our analytical solution for FFs are correct. In the next work, this method is employed to yield us the initial inputs via global fit to experimental data.

III. PION, KAON AND PROTON FRAGMENTATION FUNCTIONS

In this section, we present the results of partons fragmentation functions of pion, kaon and proton. As we did in the last sections, we can calculate the non-singlet, singlet and gluon Fragmentation Functions using analytical solution of DGLAP evolution equations in Laplace space (s, τ) . Then, with an inverse laplace transform the valence, singlet and gluon Fragmentation Functions in (z, Q^2) space are obtained. In this connection we need to use the flavor symmetries between different kinds of fragmentation functions in pion, kaon or proton at scale of Q^2 as it follows: [12]:

$$\begin{aligned} D_u^{\pi^+}(z, Q^2) &= D_d^{\pi^+}(z, Q^2) \neq D_s^{\pi^+}(z, Q^2) \\ D_u^{\pi^+}(z, Q^2) &= D_d^{\pi^+}(z, Q^2) \\ D_s^{\pi^+}(z, Q^2) &= D_{\bar{s}}^{\pi^+}(z, Q^2) \\ D_c^{\pi^+}(z, Q^2) &= D_{\bar{c}}^{\pi^+}(z, Q^2) \\ D_b^{\pi^+}(z, Q^2) &= D_{\bar{b}}^{\pi^+}(z, Q^2) \end{aligned} \quad (33)$$

$$\begin{aligned} D_u^{K^+}(z, Q^2) &\neq D_d^{K^+}(z, Q^2) \neq D_s^{K^+}(z, Q^2) \\ D_d^{K^+}(z, Q^2) &= D_{\bar{d}}^{K^+}(z, Q^2) \\ D_c^{K^+}(z, Q^2) &= D_{\bar{c}}^{K^+}(z, Q^2) \\ D_b^{K^+}(z, Q^2) &= D_{\bar{b}}^{K^+}(z, Q^2) \end{aligned} \quad (34)$$

$$\begin{aligned}
D_u^{p+}(z, Q^2) &\neq 2 D_d^{p+}(z, Q^2) \\
D_u^{p+}(z, Q^2) &\neq D_{\bar{d}}^{p+}(z, Q^2) \neq D_s^{p+}(z, Q^2) \\
D_s^{p+}(z, Q^2) &= D_{\bar{s}}^{p+}(z, Q^2) \\
D_c^{p+}(z, Q^2) &= D_{\bar{c}}^{p+}(z, Q^2) \\
D_b^{p+}(z, Q^2) &= D_{\bar{b}}^{p+}(z, Q^2) \quad (35)
\end{aligned}$$

A. Symmetry breaking in the sea quarks Fragmentation Functions

The total sea quarks fragmentation function is calculated as follows

$$D_s(z, Q^2) - D_{ns}(z, Q^2) = D_{\bar{q}}(z, Q^2) \quad (36)$$

Where $D_{\bar{q}}(z, Q^2)$ is

$$\begin{aligned}
D_{\bar{q}}(z, Q^2) &= 2D_{\bar{u}}(z, Q^2) + 2D_{\bar{d}}(z, Q^2) + 2D_s(z, Q^2) \\
&+ 2D_c(z, Q^2) + 2D_b(z, Q^2), \quad (37)
\end{aligned}$$

Now to investigate the symmetry breaking of sea quarks fragmentation functions we use the fact that heavier sea quarks can produce hadrons with higher probability. Therefore, the fraction of different kind of sea quarks can be proportional to their mass ratio. For example we have $\frac{D_{\bar{u}}}{D_c} \simeq \frac{m_u}{m_c}$. As an example, if we want to calculate the c quark fragmentation function, we have

$$\begin{aligned}
D_{\bar{q}}(z, Q^2) &= 2\frac{m_u}{m_c}D_c(z, Q^2) + 2\frac{m_d}{m_c}D_c(z, Q^2) \\
&+ 2\frac{m_s}{m_c}D_c(z, Q^2) + 2D_c(z, Q^2) + 2\frac{m_b}{m_c}D_c(z, Q^2), \quad (38)
\end{aligned}$$

$$D_c(z, Q^2) \simeq \frac{D_{\bar{q}}(z, Q^2)}{(2\frac{m_u}{m_c} + 2\frac{m_d}{m_c} + 2\frac{m_s}{m_c} + 2 + 2\frac{m_b}{m_c})} \quad (39)$$

This leads to the following general relation:

$$D_{quark}(z, Q^2) = \frac{D_{\bar{q}}(z, Q^2)}{B^A}, \quad (40)$$

where B is the mass ratio and it is constant parameter for each kind of sea quark. The free parameter A should be extracted from experimental data, however we have used HKNS code for extracting the sea quarks FFs to be sure about our analytical solutions. The results are listed in Table 1. The results for all fragmentation functions for pion, kaon and proton at $Q^2 = M_z^2$ are shown in figures 1, 2 and 3 respectively. We also compared our FFs with those from global fits of HKNS, AKK and DSS groups [12–16]. They is good agreement between them. The results show that our analytical solutions for DGLAP

quarks	A	B
\bar{u}	0.25	4651
d	0.25	2325.5
s	0.45	107.33
c	0.95	8.7893
b	2.1	2.6577

Table I: Parameters A and B in the sea quark fragmentation functions.

evolution equations are correct and these solutions are correctly used to calculate the Fragmentation Functions. In the next section we will calculate total Fragmentation Functions of pion, kaon and proton to test our calculations with experimental data.

IV. TOTAL FRAGMENTATION FUNCTIONS OF PION, KAON AND PROTON

In this section we intend to calculate the total hadron fragmentation function for pion, kaon and proton. We use FFs obtained in the previous section at $Q^2 = M_z^2$. The experiments showed that at this value of Q^2 , the interaction between electron - positron accurse via weak interaction. In this region the total hadron fragmentation function is given as follows [20, 21]

$$\frac{1}{\sigma_{tot}} \frac{d\sigma^h}{dz} = F^H(z, Q^2) = \frac{1}{\sum_q \hat{e}_q^2} [2F_1^H(z, Q^2) + F_L^H(z, Q^2)] \quad (41)$$

where we have

$$\begin{aligned}
2F_1(z, Q^2) &= \sum_q \hat{e}^2 [D_q^H + D_{\bar{q}}^H](z, Q^2) \\
&+ \frac{\alpha_s}{2\pi} [C_q^1 \otimes (D_q^H + D_{\bar{q}}^H) + C_g^1 \otimes D_g^H](z, Q^2), \quad (42)
\end{aligned}$$

$$F_L^H(z, Q^2) = \frac{\alpha_s}{2\pi} \sum_q \hat{e}^2 [C_q^L \otimes (D_q^H + D_{\bar{q}}^H) + C_g^L \otimes D_g^H], \quad (43)$$

and \hat{e}_q^2 is the Electroweak charge that is defined as

$$\hat{e}_q^2 = e_q^2 - 2e_q \chi_1(Q^2) V_e V_q + \chi_2(Q^2) (1 + V_e^2) (1 + V_q^2), \quad (44)$$

and the Electroweak parameters are defined as

$$\begin{aligned}
\chi_1(s) &= \frac{1}{16 \sin^2 \theta_W \cos^2 \theta_W} \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2}, \\
\chi_2(s) &= \frac{1}{256 \sin^4 \theta_W \cos^4 \theta_W} \frac{s^2}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2}. \quad (45)
\end{aligned}$$

$$\begin{aligned}
V_e &= -1 + 4 \sin^2 \theta_W, \\
V_u &= +1 - \frac{8}{3} \sin^2 \theta_W, \\
V_d &= -1 + \frac{4}{3} \sin^2 \theta_W.
\end{aligned} \tag{46}$$

The Wilson coefficients used in Eq. (42) and Eq. (43) are defined as follows [22],

$$\begin{aligned}
C_q^1(z) &= C_F \left[(1+z^2) \left(\frac{\ln(1-z)}{1-z} \right)_+ - \frac{3}{2} \frac{1}{(1-z)_+} \right. \\
&\quad \left. + 2 \frac{1+z^2}{1-z} \ln(z) + \frac{3}{2} (1-z) + \left(\frac{3}{2} \pi^2 - \frac{9}{2} \right) \delta(1-z) \right],
\end{aligned} \tag{47}$$

$$C_g^1(z) = 2C_F \left[\frac{1+(1-z)^2}{z} \ln(z^2(1-z)) - 2 \frac{1-z}{z} \right] \tag{48}$$

$$C_q^L(z) = C_F, \tag{49}$$

$$C_g^L(z) = 4C_F \frac{(1-z)}{z}. \tag{50}$$

The total fragmentation functions of pion, kaon and proton at the $Q^2 = M_z^2$ scale are shown in Fig. (4). We

compared our result with those from HKNS global fit and also with data from SLD experiment [11]. The agreement between data and our model is quite reasonable and it means our analytical solutions are correct.

V. CONCLUSIONS AND REMARKS

We utilized the Laplace transform technique to calculate the Laplace transform of splitting functions and extract the Fragmentation Functions of pion, kaon and proton at NLO approximation. This technique makes this facility that the analytical solution for the Fragmentation Functions (FFs) are obtained more strictly by using the related kernels and we can control the calculations in a better way.

We also found a simple approach to study the symmetry breaking in the sea quarks Fragmentation Functions. Our results are compared with those from global fits and also with experimental data which indicate good agreements between them.

We have also used the HKNS code for initial input fragmentation functions to be sure about our solutions for DGLAP evolution equations. In a new work, we are attempting to determine the initial input Fragmentation Functions by Laplace transform technique via a global fit. To do this, we have to use available data for total fragmentation functions and also multiplicity data.

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APPENDIX A

We present here the results for the Laplace transforms of splitting functions, denoted by $\Phi^{LO,NLO}$ and $\Theta^{LO,NLO}$ at the NLO approximation.

$$\Phi_f^{LO}(s) = 4 - \frac{8}{3} \left(\frac{1}{s+1} + \frac{1}{s+2} + 2(\psi(s+1) + \gamma_E) \right) \tag{51}$$

$$\Theta^{LO}_g(s) = \frac{16}{3} n_f \left(\frac{2}{s} - \frac{2}{s+2} + \frac{2}{s+3} \right), \tag{52}$$

$$\Theta^{LO}_f(s) = \frac{1}{s+1} - \frac{2}{s+2} + \frac{2}{s+3}, \tag{53}$$

$$\Phi^{LO}_g(s) = \frac{33-2n_f}{3} + 12 \left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} - \frac{1}{s+3} - \psi(s+1) - \gamma_E \right), \tag{54}$$

$$\begin{aligned}
\Phi_{nsq\bar{q}}^{NLO} = & C_F \left(-\frac{C_A}{2} + C_F \right) \left(\frac{2}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{4}{s+1} - \frac{\pi^2}{3(s+1)} - \frac{1.9968}{(s+2)^3} - \frac{2}{(s+2)^2} \right. \\
& + \frac{3.3246}{s+2} + \frac{3.9404}{(s+3)^3} - \frac{7.1312}{s+3} - \frac{3.602}{(s+4)^3} + \frac{5.8861}{s+4} + \frac{2.6484}{(s+5)^3} + \frac{3.9432}{s+5} - \frac{1.2696}{(s+6)^3} \\
& - \frac{14.24}{s+6} + \frac{0.2796}{(s+7)^3} + \frac{20.43}{s+7} - \frac{19.77}{s+8} + \frac{13.05}{s+9} + \frac{6.286}{s+10} + \frac{1.997}{s+11} - \frac{0.3076}{s+12} \\
& - 2 \left(\frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi(\frac{s}{2}+1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2}+1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2(s+1)} \right) \\
& - \frac{0.9984}{(s+2)^3} \left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2) \ln(16) - 2(s+2) \psi(\frac{s}{2}+1) + 2(s+1) \psi(\frac{s+1}{2}) \right. \\
& \left. + (s+2)^2 \psi'(\frac{s}{2}+1) - (s+2)^2 \psi'(\frac{s+1}{2}) \right) - \frac{1.9702}{(s+3)^3} \left(\frac{164}{(s+1)^2(s+2)^2} + \right. \\
& \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - \\
& 4(s+3) \ln(2) - 2(s+3) \psi(\frac{s}{2}+1) + 2(s+3) \psi(\frac{s+1}{2}) + (s+3)^2 \psi'(\frac{s}{2}+1) - \\
& \left. (s+3)^2 \psi'(\frac{s+1}{2}) \right) - \frac{1.801}{(s+4)^3} \left(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} \right. \\
& + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} \\
& + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + 4(s+4) \ln(2) - 2(s+4) \psi(\frac{s}{2}+1) + 2(s+4) \psi(\frac{s+1}{2}) + \\
& \left. (s+4)^2 \psi'(\frac{s}{2}+1) - (s+4)^2 \psi'(\frac{s+1}{2}) \right) - \frac{1.3242}{(s+5)^3} \left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \right. \\
& \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} - 4(s+5) \ln(2) \\
& - 2(5+s) \psi(\frac{s}{2}+1) + 2(5+s) \psi(\frac{s+1}{2}) + (s+5)^2 \psi'(\frac{s}{2}+1) - (s+5)^2 \psi'(\frac{s+1}{2}) \Big) - \\
& \frac{0.6348}{(s+6)^2} \left(\ln(16) - 2\psi(\frac{s}{2}+4) + 2\psi(\frac{s+7}{2}) + (s+6) \psi'(\frac{s}{2}+4) - (s+6) \psi'(\frac{s+7}{2}) \right) + \\
& \left. \frac{0.1398}{(s+7)^2} \left(\ln(16) + 2\psi(\frac{s}{2}+4) - 2\psi(\frac{s+9}{2}) - (s+7) \psi'(\frac{s}{2}+4) + (s+7) \psi'(\frac{s+9}{2}) \right) \right)
\end{aligned}$$

$$\begin{aligned}
\Phi_{nsqq}^{NLO} = & C_F T_f \left(-\frac{2}{3(s+1)^2} - \frac{2}{9(s+1)} - \frac{2}{3(s+2)^2} + \frac{22}{9(s+2)} + \frac{4}{3}\psi'(s+1) \right) + \\
& C_F^2 \left(\frac{5}{(s+1)^3} + \frac{5}{(s+1)^2} - \frac{5}{s+1} + \frac{5}{(s+2)^3} + \frac{3}{(s+2)^2} + \frac{5}{s+2} \right. \\
& - \frac{2}{(s+1)^2} \left(\gamma_E + \frac{1}{s+1} \psi(s+1) - (s+1)\psi'(s+2) \right) \\
& - \frac{2}{(s+2)^2} \left(\gamma_E + \frac{1}{s+2} \psi(s+2) - (s+2)\psi'(s+3) \right) \\
& + 4 \left((\psi(s+1) + \gamma_E) \psi'(s+1) - \frac{1}{2} \psi''(s+1) \right) - 3\psi'(s+1) + 4\psi''(s+1) \Big) \\
& + C_A C_F \left(-\frac{1}{(s+1)^3} + \frac{5}{6(s+1)^2} + \frac{53}{18(s+1)} + \frac{\pi^2}{6(s+1)} - \frac{1}{(s+2)^3} \right. \\
& + \frac{5}{6(s+2)^2} - \frac{187}{18(s+2)} + \frac{\pi^2}{6(s+2)} - \frac{67}{9} (\psi(s+1) + \gamma_E) + \frac{1}{3} \pi^2 \\
& \left. (\psi(s+1) + \gamma_E) + 2 \left(\frac{67}{18} - \frac{\pi^2}{6} \right) (\psi(s+1) + \gamma_E) - \frac{11}{3} \psi'(s+1) - \psi''(s+1) \right)
\end{aligned}$$

$$\begin{aligned}
\Phi_q^{NLO} = & C_F T_f \left(-\frac{40}{9s} + \frac{4}{(s+1)^3} + \frac{28}{3(s+1)^2} - \frac{146}{9(s+1)} + \frac{4}{(s+2)^3} + \frac{52}{3(s+2)^2} + \frac{94}{9(s+2)} + \right. \\
& \frac{16}{3(s+3)^2} + \frac{112}{9(s+3)} + \frac{4}{3}\psi'(s+1) \Big) + C_F^2 \left(\frac{7}{(s+1)^3} + \frac{3}{(s+1)^2} - \frac{1}{s+1} - \frac{\pi^2}{3(s+1)} + \right. \\
& \frac{3.0032}{(s+2)^3} + \frac{1}{(s+2)^2} + \frac{8.3246}{s+2} + \frac{3.9404}{(s+3)^3} - \frac{7.1312}{s+3} - \frac{3.602}{(s+4)^3} + \frac{5.886}{s+4} + \frac{2.6484}{(s+5)^3} \\
& + \frac{3.9432}{s+5} - \frac{1.2696}{(s+6)^3} - \frac{14.2478}{s+6} + \frac{0.2796}{(s+7)^3} + \frac{20.4376}{s+7} - \frac{19.7727}{s+8} + \frac{13.056}{s+9} - \frac{6.2862}{s+10} \\
& + \frac{1.9971}{s+11} - \frac{0.3075}{s+12} - \frac{8}{(s+1)^3} + \frac{2\ln(4)}{(s+1)^2} + \frac{2\psi(\frac{s}{2}+1)}{(s+1)^2} - \frac{2\psi(\frac{s+1}{2})}{(s+1)^2} - \frac{\psi'(\frac{s}{2}+1)}{s+1} + \\
& \frac{\psi'(\frac{s+1}{2})}{(s+1)^2} - \frac{0.9984}{(s+2)^3} \left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi(\frac{s}{2}+1) + \right. \\
& 2(s+2)\psi(\frac{s+1}{2}) + (s+2)^2\psi'(\frac{s}{2}+1) - (s+2)^2\psi'(\frac{s+1}{2}) \Big) - \frac{1.9702}{(s+3)^3} \left(\frac{164}{(s+1)^2(s+2)^2} \right. \\
& + \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - \\
& 4(s+3)\ln(2) - 2(s+3)\psi(\frac{s}{2}+1) + 2(s+3)\psi(\frac{s+1}{2}) + (s+3)^2\psi'(\frac{s}{2}+1) - (s+3)^2\psi'(\frac{s+1}{2}) \Big) \\
& - \frac{1.801}{(s+4)^3} \left(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \right. \\
& \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + \\
& 4(s+4)\ln(2) - 2(s+4)\psi(\frac{s}{2}+1) + 2(s+4)\psi(\frac{s+1}{2}) + (s+4)^2\psi'(\frac{s}{2}+1) - (s+4)^2\psi'(\frac{s+1}{2}) \Big) \\
& - \frac{1.3242}{(s+5)^3} \left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \right. \\
& \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} - 4(s+5)\ln(2) - 2(s+5)\psi(\frac{s}{2}+1) + 2(s+5)\psi(\frac{s+1}{2}) \\
& + (s+5)^2\psi'(\frac{s}{2}+1) - (s+5)^2\psi'(\frac{s+1}{2}) \Big) - \frac{2}{(s+1)^2}(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2)) \\
& - \frac{2}{(s+2)^2}(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3)) - \frac{0.6348}{(s+6)^2} \left(\ln(16) - 2\psi(\frac{s}{2}+4) + \right. \\
& 2\psi(\frac{s+7}{2}) + (s+6)\psi'(\frac{s}{2}+4) - (s+6)\psi'(\frac{s+7}{2}) \Big) + \frac{0.1398}{(s+7)^2} \left(\ln(16) + 2\psi(\frac{s}{2}+4) - \right. \\
& 2\psi(\frac{s+9}{2}) - (s+7)\psi'(\frac{s}{2}+4) + (s+7)\psi'(\frac{s+9}{2}) \Big) + \\
& 4 \left((\psi(s+1) + \gamma_E)\psi'(s+1) - \frac{1}{2}\psi''(s+1) \right) - 3\psi'(s+1) + 4\psi''(s+1) \Big) +
\end{aligned}$$

$$\begin{aligned}
& C_A C_F \left(\frac{2}{(s+1)^3} + \frac{11}{6(s+1)^2} + \frac{17}{18(s+1)} + \frac{\pi^2}{3(s+1)} - \frac{0.0016}{(s+2)^3} + \frac{11}{6(s+2)^2} - \frac{10.4062}{s+2} \right. \\
& - \frac{1.9702}{(s+3)^3} + \frac{3.5656}{s+3} + \frac{1.801}{(s+4)^3} - \frac{2.9430}{s+4} - \frac{1.3242}{(s+5)^3} - \frac{1.9716}{s+5} + \frac{0.6348}{(s+6)^3} + \frac{7.1239}{s+6} \\
& - \frac{0.1398}{(s+7)^3} - \frac{10.2188}{s+7} + \frac{9.8863}{s+8} - \frac{6.5284}{s+9} + \frac{3.1431}{s+10} - \frac{0.9985}{s+11} + \frac{0.1537}{s+12} - \\
& \frac{67(\psi(s+1) + \gamma_E)}{9} + \frac{1}{3}\pi^2(\psi(s+1) + \gamma_E) + 2 \left(\frac{67}{18} - \frac{\pi^2}{6} \right) (\psi(s+1) + \gamma_E) - \frac{\ln(4)}{(s+1)^2} - \\
& \frac{\psi(\frac{s}{2} + 1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2} + 1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2s+2} + \frac{0.4992}{(s+2)^3} \left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) \right. \\
& - 2(s+2)\psi(\frac{s}{2} + 1) + 2(s+2)\psi(\frac{s+1}{2}) + (s+2)^2\psi'(\frac{s}{2} + 1) - (s+2)^2\psi'(\frac{s+1}{2}) \Big) + \\
& \frac{0.9851}{(s+3)^3} \left(\frac{164}{(s+1)^2(s+2)^2} + \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \right. \\
& \frac{8s^4}{(s+1)^2(s+2)^2} - 4(s+3)\ln(2) - 2(s+3)\psi(\frac{s}{2} + 1) + 2(s+3)\psi(\frac{s+1}{2}) + \\
& (s+3)^2\psi'(\frac{s}{2} + 1) - (s+3)^2\psi'(\frac{s+1}{2}) \Big) + \frac{0.9005}{(s+4)^3} \\
& \left(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \right. \\
& \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + \\
& 4(s+4)\ln(2) - 2(s+4)\psi(\frac{s}{2} + 1) + 2(s+4)\psi(\frac{s+1}{2}) + (s+4)^2\psi'(\frac{s}{2} + 1) - (s+4)^2\psi'(\frac{s+1}{2}) \Big) \\
& + \frac{0.6621}{(s+5)^3} \left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \right. \\
& \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} - 4(s+5)\ln(2) - 2(s+5)\psi(\frac{s}{2} + 1) + 2(s+5)\psi(\frac{s+1}{2}) + \\
& (s+5)^2\psi'(\frac{s}{2} + 1) - (s+5)^2\psi'(\frac{s+1}{2}) \Big) + \frac{0.3174}{(s+6)^2} \left(\ln(16) - 2\psi(\frac{s}{2} + 4) + 2\psi(\frac{s+7}{2}) + \right. \\
& (s+6)\psi'(\frac{s}{2} + 4) - (s+6)\psi'(\frac{s+7}{2}) \Big) - \frac{0.0699}{(s+7)^2} \left(\ln(16) + 2\psi(\frac{s}{2} + 4) - 2\psi(\frac{s+9}{2}) - \right. \\
& (s+7)\psi'(\frac{s}{2} + 4) + (s+7)\psi'(\frac{s+9}{2}) \Big) - \frac{11}{3}\psi'(s+1) - \psi''(s+1) \Big)
\end{aligned}$$

$$\begin{aligned}
\Theta_q^{NLO} = & T_f^2 \left(\frac{8}{3(s+1)^2} - \frac{40}{9(s+1)} - \frac{16}{3(s+2)^2} + \frac{32}{9(s+2)} + \frac{16}{3(s+3)^2} - \frac{32}{9(s+3)} + \right. \\
& \left. \frac{8(\psi(s+2) + \gamma_E)}{3(s+1)} - \frac{16(\psi(s+3) + \gamma_E)}{3(s+2)} + \frac{16(\psi(s+4) + \gamma_E)}{3(s+3)} \right) + \\
& C_A T_f \left(-\frac{40}{9s} + \frac{4}{(s+1)^3} + \frac{8}{3(s+1)^2} + \frac{26}{9(s+1)} + \frac{24}{(s+2)^3} + \frac{68}{3(s+2)^2} - \frac{33.231}{s+2} \right. \\
& - \frac{4\pi^2}{3(s+2)} + \frac{8}{3(s+3)^2} + \frac{96.875}{s+3} - \frac{67.644}{s+4} + \frac{83.04}{s+5} - \frac{82.976}{s+6} + \frac{56.16}{s+7} - \frac{22}{s+8} \\
& - \frac{22(\psi(s+2) + \gamma_E)}{3(s+1)} + \frac{20(\psi(s+3) + \gamma_E)}{3(s+2)} - \frac{20(\psi(s+4) + \gamma_E)}{3(s+3)} \\
& - 2 \left(\frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi(\frac{s}{2} + 1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2} + 1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2(s+1)} \right) + \\
& \frac{2}{(s+2)^3} \left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2) \ln(16) - 2(s+2)\psi(\frac{s}{2} + 1) + 2(s+2)\psi(\frac{s+1}{2}) \right. \\
& \left. + (s+2)^2\psi'(\frac{s}{2} + 1) - (s+2)^2\psi'(\frac{s+1}{2}) \right) - \frac{2}{(s+3)^3} \left(\frac{164}{(s+1)^2(s+2)^2} + \frac{284s}{(s+1)^2(s+2)^2} \right. \\
& \left. + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - 4(s+3) \ln(2) - 2(s+3)\psi(\frac{s}{2} + 1) + \right. \\
& \left. 2(s+3)\psi(\frac{s+1}{2}) + (s+3)^2\psi'(\frac{s}{2} + 1) - (s+3)^2\psi'(\frac{s+1}{2}) \right) + \frac{2(\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2))}{6s+6} \\
& - \frac{8}{(s+1)^2} \left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2) \right) - \frac{4(\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3))}{6s+12} \\
& + \frac{16}{(s+2)^2} \left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) + \frac{4(\pi^2 + 6(\psi(s+4) + \gamma_E)^2 - 6\psi'(s+4))}{6s+18} \\
& - \frac{16}{(s+3)^2} \left(\gamma_E + \frac{1}{s+3} + \psi(s+3) - (s+3)\psi'(s+4) \right) \Big) + \\
& C_F T_f \left(-\frac{2}{(s+1)^3} + \frac{7}{(s+1)^2} - \frac{12}{s+1} - \frac{2\pi^2}{3(s+1)} + \frac{4}{(s+2)^3} - \frac{8}{(s+2)^2} + \frac{39.16}{s+2} + \frac{4\pi^2}{3(s+2)} - \frac{8}{(s+3)^3} \right. \\
& - \frac{65.856}{s+3} - \frac{4\pi^2}{3(s+3)} + \frac{77.872}{s+4} - \frac{81.216}{s+5} + \frac{80.128}{s+6} - \frac{51.968}{s+7} + \frac{17.6}{s+8} + \frac{2(\psi(s+1) + \gamma_E)}{s+1} + \\
& \frac{4(\psi(s+2) + \gamma_E)}{s+1} - \frac{4(\psi(s+2) + \gamma_E)}{s+2} + \frac{4(\psi(s+3) + \gamma_E)}{s+3} - \frac{2(\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2))}{6s+6} \\
& + \frac{12}{(s+1)^2} \left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2) \right) + \frac{4(\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3))}{6s+12} \\
& - \frac{24}{(s+2)^2} \left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) - \frac{4(\pi^2 + 6(\psi(s+4) + \gamma_E)^2 - 6\psi'(s+4))}{6s+18} + \\
& \left. \frac{24}{(s+3)^2} \left(\gamma_E + \frac{1}{s+3} + \psi(s+3) - (s+3)\psi'(s+4) \right) \right)
\end{aligned}$$

$$\begin{aligned}
\Theta_g^{NLO} = & C_F^2 \left(\frac{2}{(s+1)^3} + \frac{8}{(s+1)^2} - \frac{16.66}{s+1} - \frac{1}{(s+2)^3} - \frac{1}{2(s+2)^2} + \frac{34.196}{s+2} - \frac{40.096}{s+3} + \right. \\
& \frac{42.432}{s+4} - \frac{35.224}{s+5} + \frac{17.392}{s+6} - \frac{4.4}{s+7} - \frac{2(\psi(s+3) + \gamma_E)}{s+2} + \\
& \frac{1}{3s} (\pi^2 + 6(\psi(s+1) + \gamma_E)^2 - 6\psi'(s+1)) - \frac{8}{s^3} (1 + s\gamma_E + s(\psi(s) - s\psi'(s+1))) - \\
& \frac{2}{6s+6} (\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2)) + \frac{8}{(s+1)^2} \left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - \right. \\
& (s+1)\psi'(s+2)) + \frac{1}{6s+12} (\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3)) - \\
& \left. \frac{4}{(s+2)^2} \left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) \right) + \\
& C_A C_F \left(-\frac{4}{s^3} + \frac{6}{s^2} + \frac{17}{9s} - \frac{2\pi^2}{3s} - \frac{8}{(s+1)^2} + \frac{25.2}{s+1} - \frac{4}{(s+2)^3} - \frac{9}{(s+2)^2} - \frac{23.27}{s+2} - \right. \\
& \frac{\pi^2}{3(s+2)} - \frac{8}{3(s+3)^2} + \frac{35.99}{s+3} - \frac{41.046}{s+4} + \frac{35.01}{s+5} - \frac{17.444}{s+6} + \frac{3.3}{s+7} + \frac{2(\psi(s+3) + \gamma_E)}{s+2} \\
& + \frac{1}{s^2} \left(\ln(16) - 2\psi\left(\frac{s}{2} + 1\right) + 2\psi\left(\frac{s+1}{2}\right) + s\psi'\left(\frac{s}{2} + 1\right) - s\psi'\left(\frac{s+1}{2}\right) \right) - \\
& 2 \left(\frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi(\frac{s}{2} + 1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2} + 1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2s+2} \right) + \\
& \frac{1}{2(s+2)^3} \left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi\left(\frac{s}{2} + 1\right) + 2(s+2)\psi\left(\frac{s+1}{2}\right) + \right. \\
& (s+2)^2\psi'\left(\frac{s}{2} + 1\right) - (s+2)^2\psi'\left(\frac{s+1}{2}\right) \left. \right) - \frac{1}{3s} \\
& \left(\pi^2 + 6(\psi(s+1) + \gamma_E)^2 - 6\psi'(s+1) \right) + \frac{12}{s^3} (1 + s\gamma_E + s(\psi(s) - s\psi'(s+1))) + \\
& \frac{2}{6s+6} (\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2)) - \\
& \frac{12}{(s+1)^2} \left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2) \right) - \\
& \frac{1}{6s+12} (\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3)) + \\
& \left. \frac{6}{(s+2)^2} \left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) \right)
\end{aligned}$$

$$\begin{aligned}
\Phi_g^{NLO} = & C_F T_f \left(-\frac{16}{3s^2} + \frac{92}{9s} + \frac{4}{(s+1)^3} - \frac{10}{(s+1)^2} - \frac{4}{s+1} + \frac{4}{(s+2)^3} - \frac{14}{(s+2)^2} + \frac{12}{s+2} - \frac{16}{3(s+3)^2} - \frac{164}{9(s+3)} \right) \\
& + C_A T_f \left(\frac{8}{3s^2} - \frac{46}{9s} - \frac{4}{(s+1)^2} + \frac{58}{9(s+1)} + \frac{4}{(s+2)^2} - \frac{38}{9(s+2)} - \frac{8}{3(s+3)^2} + \frac{46}{9(s+3)} \right. \\
& \left. + \frac{8}{3}\psi'(s+1) \right) + C_A^2 \left(-\frac{8}{s^3} + \frac{22}{3s^2} + \frac{2}{(s+1)^3} + \frac{11}{(s+1)^2} + \frac{4.4407}{s+1} - \frac{17.9984}{(s+2)^3} + \frac{1}{(s+2)^2} - \right. \\
& \frac{6.9024}{s+2} - \frac{\pi^2}{3(s+2)} + \frac{5.9702}{(s+3)^3} + \frac{22}{3(s+3)^2} - \frac{6.7917}{s+3} + \frac{\pi^2}{3(s+3)} - \frac{1.801}{(s+4)^3} - \frac{3.5389}{s+4} + \\
& \frac{1.3242}{(s+5)^3} + \frac{1.2736}{s+5} - \frac{0.6348}{(s+6)^3} - \frac{5.6479}{s+6} + \frac{0.1398}{(s+7)^3} + \frac{9.2228}{s+7} - \frac{7.6863}{s+8} + \frac{6.5284}{s+9} - \frac{3.1431}{s+10} \\
& + \frac{0.9985}{s+11} - \frac{0.1537}{s+12} - \frac{67(\psi(s+1) + \gamma_E)}{9} + \frac{1}{3}\pi^2(\psi(s+1) + \gamma_E) - \frac{1}{s^2} \left(\ln(16) - 2\psi\left(\frac{s}{2} + 1\right) + \right. \\
& 2\psi\left(\frac{s+1}{2}\right) + s\psi'\left(\frac{s}{2} + 1\right) - s\psi'\left(\frac{s+1}{2}\right) \Big) + 2 \left(\frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi\left(\frac{s}{2} + 1\right)}{(s+1)^2} + \frac{\psi\left(\frac{s+1}{2}\right)}{(s+1)^2} + \right. \\
& \frac{\psi'\left(\frac{s}{2} + 1\right)}{2s+2} - \frac{\psi'\left(\frac{s+1}{2}\right)}{2s+2} \Big) - \frac{1.9992}{(s+2)^3} \left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi\left(\frac{s}{2} + 1\right) \right. \\
& + 2(s+2)\psi\left(\frac{s+1}{2}\right) + (s+2)^2\psi'\left(\frac{s}{2} + 1\right) - (s+2)^2\psi'\left(\frac{s+1}{2}\right) \Big) + \frac{0.0149}{(s+1)^3} \left(\frac{164}{(s+1)^2(s+2)^2} \right. \\
& + \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - 4(s+3)\ln(2) - \\
& 2(s+3)\psi\left(\frac{s}{2} + 1\right) + 2(s+3)\psi\left(\frac{s+1}{2}\right) + (s+3)^2\psi'\left(\frac{s}{2} + 1\right) - (s+3)^2\psi'\left(\frac{s+1}{2}\right) \Big) - \frac{0.9005}{(s+4)^3} \\
& \left(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} \right. \\
& + \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + \\
& 4(s+4)\ln(2) - 2(s+4)\psi\left(\frac{s}{2} + 1\right) + 2(s+4)\psi\left(\frac{s+1}{2}\right) + (s+4)^2\psi'\left(\frac{s}{2} + 1\right) - (s+4)^2\psi'\left(\frac{s+1}{2}\right) \Big) \\
& - \frac{0.6621}{(s+5)^3} \left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \right. \\
& + \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \\
& + \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \\
& + \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \\
& \left. - 4(s+5)\ln(2) - 2(s+5)\psi\left(\frac{s}{2} + 1\right) + 2(s+5)\psi\left(\frac{s+1}{2}\right) + (s+5)^2\psi'\left(\frac{s}{2} + 1\right) - (s+5)^2\psi'\left(\frac{s+1}{2}\right) \right) \\
& + \frac{4}{s^3}(1 + \gamma_E s + s(\psi(s) - s\psi'(s+1))) - \frac{8}{(s+1)^2}(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2)) + \\
& \frac{4}{(s+2)^2}(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3)) - \frac{4}{(s+3)^2}(\gamma_E + \frac{1}{s+3} + \psi(s+3) - (s+3)\psi'(s+4)) \Big) \\
& - \frac{0.3174}{(s+6)^2} \left(\ln(16) - 2\psi\left(\frac{s}{2} + 4\right) + 2\psi\left(\frac{s+7}{2}\right) + (s+6)\psi'\left(\frac{s}{2} + 4\right) - (s+6)\psi'\left(\frac{s+7}{2}\right) \right) + \\
& \frac{0.0699}{(s+7)^2} \left(\ln(16) + 2\psi\left(\frac{s}{2} + 4\right) - 2\psi\left(\frac{s+9}{2}\right) - (s+7)\psi'\left(\frac{s}{2} + 4\right) + (s+7)\psi'\left(\frac{s+9}{2}\right) \right) \\
& + 4 \left((\psi(s+1) + \gamma_E)\psi'(s+1) - \frac{1}{2}\psi''(s+1) \right) - \frac{22}{3}\psi'(s+1) + 3\psi''(s+1) \Big)
\end{aligned}$$

$$\begin{aligned}
C_q^L &= \frac{C_F}{s+1} \\
C_g^L &= 4C_F \left(\frac{1}{s} - \frac{1}{s+1} \right) \\
C_q^1 &= \frac{-9}{2} + \frac{2\pi^2}{3} + \frac{2}{3(s+1)} - \frac{2}{3(s+2)} - 2\psi'(s+1) - 2\psi'(s+3) \\
C_g^1 &= 2C_F \left(-\frac{2}{s} + \frac{2}{s+1} - \frac{2(2+s(\psi(s+1)+\gamma_E))}{s^2} + \frac{2(2+(1+s)(\psi(s+1)+\gamma_E))}{(s+1)^2} \right. \\
&\quad \left. + \frac{2+(s+2)(\psi(s+3)+\gamma_E)}{(s+2)^2} \right)
\end{aligned}$$

APPENDIX B

We bring in below the coefficients of singlet and gluon distributions of Eq. (32), k_{ij} , in the Laplace transformed, s space:

$$\begin{aligned}
k_{ff} &= e^{\frac{1}{2}\tau(-2b_1+\Phi_f+\Phi_g-R)} (b_1(a_1\Theta_f\Theta_g(-\Phi_f-\Phi_g+R) + a_1e^{\tau R}\Theta_f\Theta_g(\Phi_f+\Phi_g+R) \\
&\quad + b_1^2e^{b_1\tau}((1-e^{\tau R})(\Phi_f-\Phi_g)-R-e^{\tau R}R) + e^{\tau(b_1+R)}(\Phi_f^3+\Phi_f^2(-3\Phi_g+R) + \\
&\quad (\Phi_g^2-(-4+a_1)\Theta_f\Theta_g)(-\Phi_g+R) + \Phi_f(3\Phi_g^2+(4+a_1)\Theta_f\Theta_g-2\Phi_gR)) + \\
&\quad e^{b_1\tau}(-\Phi_f^3+(\Phi_g^2-(-4+a_1)\Theta_f\Theta_g)(\Phi_g+R) + \Phi_f^2(3\Phi_g+R) - \Phi_f(3\Phi_g^2+ \\
&\quad (4+a_1)\Theta_f\Theta_g+2\Phi_gR))) + 4a_1e^{\frac{1}{2}\tau(b_1+R)}(R(-b_1^2\Phi_f+\Phi_f(\Phi_f-\Phi_g)^2 + \\
&\quad (3\Phi_f-\Phi_g)\Theta_f\Theta_g \cosh(\frac{1}{2}\tau R) \sinh(\frac{b_1\tau}{2}) + (b_1^2\Theta_f\Theta_g \cosh(\frac{b_1\tau}{2}) - (b_1^2 - \\
&\quad (\Phi_f-\Phi_g)^2 - 4\Theta_f\Theta_g)(\Phi_f^2-\Phi_f\Phi_g+\Theta_f\Theta_g) \\
&\quad \sinh(\frac{b_1\tau}{2})) \sinh(\frac{1}{2}\tau R))))/(2b_1R(-b_1^2+R^2))
\end{aligned}$$

$$\begin{aligned}
k_{gf} &= (e^{\frac{1}{2}\tau(-2b_1+\Phi_f+\Phi_g-R)} (b_1\Theta_g(-2b_1^2e^{b_1\tau}(-1+e^{\tau R}) + e^{\tau(b_1+R)}(2\Phi_f^2-(4+a_1) \\
&\quad \Phi_f\Phi_g + (2+a_1)\Phi_g^2 + 2(4+a_1)\Theta_f\Theta_g - a_1\Phi_gR) - e^{b_1\tau}(2\Phi_f^2-(4+a_1) \\
&\quad \Phi_f\Phi_g + (2+a_1)\Phi_g^2 + 2(4+a_1)\Theta_f\Theta_g + a_1\Phi_gR) + a_1(-2\Theta_f\Theta_g + \Phi_g \\
&\quad (\Phi_f-\Phi_g+R)) + a_1e^{\tau R}(2\Theta_f\Theta_g + \Phi_g(-\Phi_f+\Phi_g+R))) + 4a_1e^{\frac{1}{2}\tau(b_1+R)} \\
&\quad \Theta_g((-b_1^2+\Phi_f^2-\Phi_f\Phi_g+2\Theta_f\Theta_g)R \cosh(\frac{1}{2}\tau R) \sinh(\frac{b_1\tau}{2}) + (b_1^2\Phi_g \cosh(\frac{b_1\tau}{2}) \\
&\quad + \Phi_f(-b_1^2+R^2) \sinh(\frac{b_1\tau}{2})) \\
&\quad \sinh(\frac{1}{2}\tau R))))/(2b_1R(-b_1^2+R^2))
\end{aligned}$$

$$\begin{aligned}
k_{fg} &= (2e^{\frac{1}{2}(-b_1+\Phi_f+\Phi_g)\tau} \Theta_f(-a_1((b_1+\Phi_f-\Phi_g)(b_1+\Phi_g) - 2\Theta_f\Theta_g)R \cosh(\frac{1}{2}\tau R) \\
&\quad \sinh(\frac{b_1\tau}{2}) + (b_1(-(b_1+\Phi_f-\Phi_g)(b_1-(1+a_1)\Phi_f+\Phi_g) + 2(2+a_1)\Theta_f\Theta_g) \\
&\quad \cosh(\frac{b_1\tau}{2}) - (b_1+a_1\Phi_g)(b_1^2-(\Phi_f-\Phi_g)^2 - 4\Theta_f\Theta_g) \sinh(\frac{b_1\tau}{2})) \\
&\quad \sinh(\frac{1}{2}\tau R)))/(b_1R(-b_1^2+R^2))
\end{aligned}$$

$$\begin{aligned}
k_{gg} = & (e^{\frac{1}{2}\tau(-2b_1+\Phi_f+\Phi_g-R)}(b_1(a_1\Theta_f\Theta_g(-\Phi_f-\Phi_g+R)+a_1e^{\tau R}\Theta_f\Theta_g \\
& (\Phi_f+\Phi_g+R)+b_1^2e^{b_1\tau}((-1+e^{\tau R})\Phi_f+\Phi_g-R-e^{\tau R}(\Phi_g+R))) + \\
& e^{b_1\tau}(\Phi_f^3-\Phi_g^3-(4+a_1)\Phi_g\Theta_f\Theta_g+\Phi_g^2R-(-4+a_1)\Theta_f\Theta_gR+ \\
& \Phi_f^2(-3\Phi_g+R)+\Phi_f(3\Phi_g^2-(-4+a_1)\Theta_f\Theta_g-2\Phi_gR))+e^{\tau(b_1+R)} \\
& (-\Phi_f^3+\Phi_g^3+(4+a_1)\Phi_g\Theta_f\Theta_g+\Phi_g^2R-(-4+a_1)\Theta_f\Theta_gR+\Phi_f^2 \\
& (3\Phi_g+R)+\Phi_f(-3\Phi_g^2+(-4+a_1)\Theta_f\Theta_g-2\Phi_gR))) + \\
& 4a_1e^{\frac{1}{2}\tau(b_1+R)}(R(-b_1^2\Phi_g+(\Phi_f-\Phi_g)^2\Phi_g-(\Phi_f-3\Phi_g)\Theta_f\Theta_g) \\
& \cosh(\frac{1}{2}\tau R)\sinh(\frac{b_1\tau}{2})+(b_1^2\Theta_f\Theta_g\cosh(\frac{b_1\tau}{2})+(b_1^2-(\Phi_f-\Phi_g)^2- \\
& 4\Theta_f\Theta_g)((\Phi_f-\Phi_g)\Phi_g-\Theta_f\Theta_g)\sinh(\frac{b_1\tau}{2})) \\
& \sinh(\frac{1}{2}\tau R))))/(2b_1R(-b_1^2+R^2))
\end{aligned}$$

$$R = \sqrt{(\Phi_f - \Phi_g)^2 + 4\Theta_f\Theta_g}$$

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